Math 121 2.3 Some Differentiation Formulas

Objectives

1) Find derivatives using rules (shortcuts) instead of the definition (limit).

- constant rule
- power rule
- constant multiple rule
- sum rule
- difference rule

2) Recall: derivative tells us the instantaneous rate of change and the slope of a tangent line

- write equations of tangent lines

3) Business and Economics applications of the derivative

- marginal revenue $MR(x) = R'(x)$
- marginal cost $MC(x) = C'(x)$
- marginal profit $MP(x) = P'(x)$.

* all of these are just derivatives _ rates of change at a particular production level, e.g. manufacturing x tennis raquets.

CAUTION: Notice instructions for chapter 2:

"Find the derivative (or instantaneous rate of change or slope of tangent) using the definition" means LIMITS.

"Find the derivative" means LIMITS or SHORTCUTS — but why use limits if you don't have to?

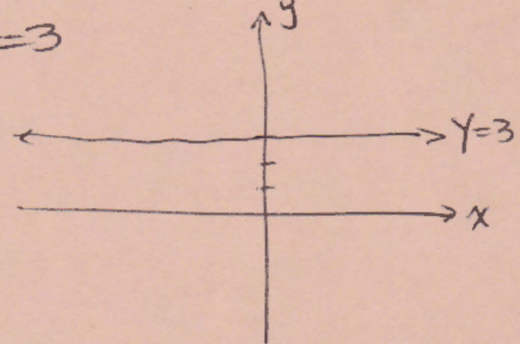
Math 121

Recall: The derivative $f'(x)$ is a function that

- is related to $f(x)$
- is a different function from $f(x)$
  (in all but one really special case)
- gives the slope of the tangent line to $f(x)$ at x.

So... if $f(x)$ is a constant...

① $f(x)=3$

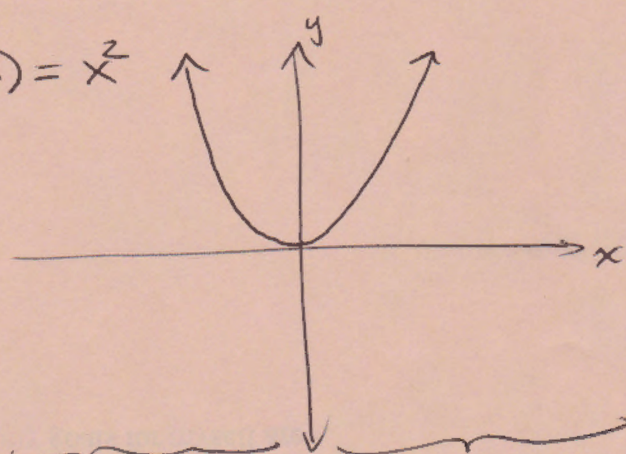


any tangent line to $y=3$
is the same as $y=3$
as has slope 0
for any value of x.

$f'(x)=0$.

$$\boxed{\text{Constant Rule} \quad \frac{d}{dx}[c] = 0}$$

② $f(x)=x^2$



tangent lines here have negative slope

tangent lines here have positive slope

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\left(\frac{2xh}{h}+\frac{h^2}{h}\right) = \lim_{h\to 0}(2x+h) = 2x = f'(x)$$

3. $f(x) = x^3$ gives $f'(x) = 3x^2$

4. $g(x) = x^4$ gives $g'(x) = 4x^3$

5. $h(x) = x^{-3}$ gives $h'(x) = -3x^{-4}$ ← The exponent is always subtract 1.

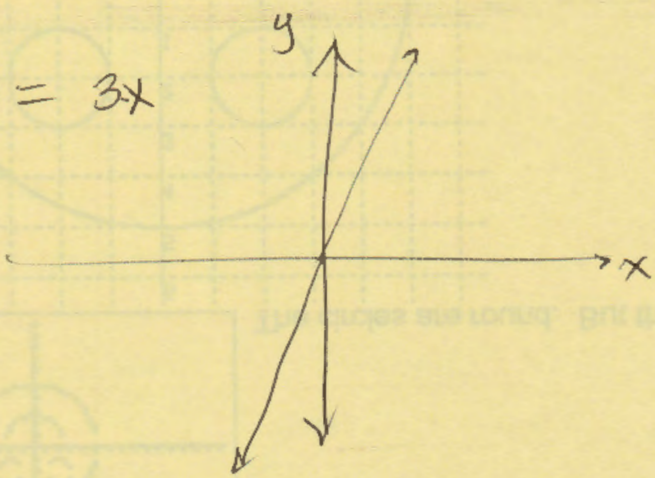
6. $k(x) = x^{1/2}$ gives $k'(x) = \frac{1}{2}x^{-1/2}$

7. $m(x) = x$ gives $m'(x) = 1 \cdot x^0 = 1$

$$\boxed{\text{Power Rule} \quad \frac{d}{dx}x^n = n \cdot x^{n-1}}$$

See Math 250 notes for 3.3 for more details of why

7. $f(x) = 3x$



line $y=3x$ has slope 3 for all x.

tangent line to $y=3x$ is the same $y=3x$, slope 3 for all x.

8. $g(x) = 3x^2$ gives $g'(x) = \underbrace{3}_{\text{constant multiple}} \cdot \underbrace{2x}_{\text{power rule}} = 6x$

$$\boxed{\text{Constant Multiple Rule} \quad \frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot f'(x)}$$

By the defn

$$\frac{d}{dx}\left[c \cdot f(x)\right]$$
$$= \lim_{h \to 0} \frac{c \cdot f(x+h) - c f(x)}{h}$$
$$= \lim_{h \to 0} c \cdot \left[\frac{f(x+h) - f(x)}{h}\right]$$
$$= c \cdot \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right]$$
$$= c \cdot f'(x)$$

9. $f(x) = 3x + 5 \Longrightarrow f'(x) = 3$

10. $g(x) = 3x^2 + 6 \Longrightarrow g'(x) = 6x$

Sum Rule

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

Difference Rule

$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$$

$\boxed{\text{* CAUTION *}}$

We cannot separate $f(x) \cdot g(x)$ multiplication or $\frac{f(x)}{g(x)}$ division. These require more complex rules.

Sum Rule by the definition: (Difference)

$$\frac{d}{dx}[f(x) \pm g(x)] = \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h}$$ substitute

$$= \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - f(x) \mp g(x)}{h}$$ dist

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) \pm g(x+h) - g(x)}{h}$$ rearrange (commutative)

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$ separate

$$= f'(x) \pm g'(x)$$

Recall:

The derivative of a function tells us

- slope of tangent line at x
- instantaneous rate of change at x.
- also called, in application, the marginal rate of change at x.

ex. Marginal Cost $MC(x) = C'(x)$ where $C(x)$ is the cost of producing x objects.

ex. Marginal Revenue $MR(x) = R'(x)$ where $R(x)$ is the revenue when x objects are produced.

ex. Marginal Profit $MP(x) = P'(x)$ where $P(x)$ is the profit $(R(x) - C(x))$ when x objects are produced.

·Practice·

Find derivatives.

① $f(x) = 12\sqrt{x}$

$f(x) = 12 \cdot x^{1/2}$ write as exponent

$f'(x) = 12 \cdot \frac{1}{2} x^{-1/2}$ constant multiple 12 and power rule

$\boxed{f'(x) = 6x^{-1/2}}$ simplify

$\boxed{f'(x) = \frac{6}{x^{1/2}}}$ positive exp

$\boxed{f'(x) = \frac{6}{\sqrt{x}}}$ radical

② $f(x) = \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$

power rules

$f'(x) = \frac{1}{24} \cdot 4x^3 + \frac{1}{6} \cdot 3x^2 + \frac{1}{2} \cdot 2x + 1x + 0$

constant multiples

constant term

$\boxed{f'(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1}$

③ $f(x) = \frac{4x^3 - 2x^2}{x}$

simplify first

$f(x) = \frac{4x^3}{x} - \frac{2x^2}{x}$

$f(x) = 4x^2 - 2x$

$f'(x) = 4 \cdot 2x - 2 \cdot 1$

$\boxed{f'(x) = 8x - 2}$

④ $f(x) = \dfrac{9}{2\sqrt[3]{x^2}} - 16\sqrt{x^5} - 14$

a) find $f'(x)$

b) find equation of tangent line to $f(x)$ at $x=1$.

a) Rewrite as power rules → exponents must be on bases in the numerators.

$$f(x) = \frac{9}{2}x^{-2/3} - 16x^{5/2} - 14$$

Differentiate

$$f'(x) = \frac{9}{2}\cdot\left(\frac{-2}{3}\right)x^{-2/3-1} - 16\cdot\frac{5}{2}\cdot x^{5/2-1} - 0$$

$$\boxed{f'(x) = -3x^{-5/3} - 40x^{3/2}}$$

b) Slope of tangent line $= f'(1) = -3(1)^{-5/3} - 40(1)^{3/2}$

$= -3 - 40$

$= -43.$

x - coordinate $= 1$ (given)

y-coordinate $= f(1)$ (using original function)

$= \dfrac{9}{2\sqrt[3]{1^2}} - 16\sqrt{1^5} - 14$

$= \dfrac{9}{2} - 16 - 14$

$= -25.5 = -\dfrac{51}{2}$

ordered pair $\left(1, -\dfrac{51}{2}\right)$

line $y - \left(-\dfrac{51}{2}\right) = -43(x-1)$ point-slope formula

$y + \dfrac{51}{2} = -43x + 43$ dist

$$\boxed{y = -43x + \frac{35}{2}}$$

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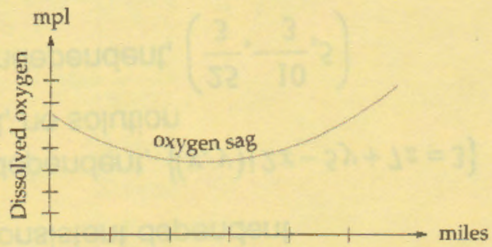
58. ENVIRONMENTAL SCIENCE: Water Quality
Downstream from a waste treatment plant the amount of dissolved oxygen in the water usually decreases for some distance (due to bacteria consuming the oxygen) and then increases (due to natural purification). A graph of the dissolved oxygen at various distances downstream looks like the curve below (known as the "oxygen sag"). The amount of dissolved oxygen is usually taken as a measure of the health of the river.



Distance downstream from treatment plant

Suppose that the amount of dissolved oxygen x miles downstream is $D(x) = 0.2x^2 - 2x + 10$ mpl (milligrams per liter) for $0 \le x \le 20$. Use this formula to find the instantaneous rate of change of the dissolved oxygen:

a. 1 mile downstream.
b. 10 miles downstream.

Interpret the signs of your answers.

"instantaneous rate of change" means "derivative"

a) and b) both require $D'(x)$.

$D(x) = 0.2x^2 - 2x + 10$

D measured in mpl

x measured in miles

$D'(x) = (0.2)2x - 2 \cdot x^0 + 0$

$D'(x) = .4x - 2$

a) $D'(1) = .4(1) - 2 = \boxed{-1.6 \text{ mpl/mi}}$

Interpret: Amount of dissolved oxygen is <u>decreasing</u> by 1.6 mpl per mile.

b) $D'(10) = .4(10) - 2 = \boxed{2 \text{ mpl/mi}}$

Amount of dissolved oxygen is <u>increasing</u> by 2 mpl per mile.